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On a New Condition Distinguishing Weyl and Lorentz Space-Times

Udo Schelb[1]

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The paper is concerned with the derivability of a Lorentz instead of only a Weyl manifold as space-time structure from postulates about free fall and light propagation. For this purpose it identifies a property distinguishing both kinds of space-times. The property is one of a particular metric of the conformal class of the Weyl manifold. viz. that in suitably chosen locally geodesic coordinates the g_{i4} components, $i = 1, 2, 3$ vanish along the time axis. Although seemingly somewhat hidden, one is led to this property in looking for a metric which can play a distinguished role. We demonstrate that for a Lorentzian manifold such a condition is always given; thus it is a necessary one. It is sufficient since for a Weyl space it has the consequence that the metric connection of the selected g is projectively equivalent to the Weyl connection. Thus, if a Weyl space-time complies with it, it is a reducible one. The results of this paper lay the ground for deriving in a second step this condition from a simple, empirically testable postulate about free-fall worldlines and "radar" measurements by light signals.

1. INTRODUCTION

1.1. The Problem

Nearly as long as the history of general relativity itself is the history of efforts to improve the understanding of its underlying space-time picture—that of a four-dimensional pseudo-Riemannian manifold (M, g). Weyl (1921) elucidated the different inherent mathematical structures in (M, g), and Reichenbach (1924, 1969) gave a constructive axiomatics for (special and general) relativity theory on the basis of some primitive notions. In this the former looked mainly from a mathematical, the latter mainly from a physical-epistemological point of view. We think that a *space-time axiomatics* which

[1] Department of Physics, University of Paderborn, 33098 Paderborn, Germany; e-mail: udo@lagrange.uni-paderborn.de.

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is a kind of combination of these two approaches is an appropriate tool for a physically more intelligible foundation of (M, g). There one starts from some assumptions, as simple and plausible as possible, about simple physical objects and tries to establish the various mathematical structures of (M, g) on these grounds. A well-known example of such a theory is that of Ehlers *et al.* (1972) (EPS); freely falling particles and light signals are used as simple physical objects, resp. primitive notions of the axiomatics. Some empirically founded and testable properties of these objects are translated into mathematical axioms which restrict the possible space-time models. The title of the original paper, "The geometry of free fall and light propagation," expresses concisely the aim of their work and of related efforts of other authors.

However, insofar as one applies only axioms which can be formulated *directly* in terms of free fall and light, the resulting geometry is not the pseudo-Riemannian, but only the more general Weyl one. Only by additional postulates of a much stronger, less direct kind could one restrict the space-time picture further to a Lorentzian manifold. On the other hand, a Weyl geometry is generally not seen as a physically acceptable space-time model, since in it time and length scales cannot be transported in a path-independent manner. Our paper aims at laying the basis for a new, more direct postulate by identifying an inconspicuous characteristic feature which marks the transition from a Weyl to a Lorentz space. Such a postulate is formulated in Schelb (1996); it is related to so-called radar distance measurements with the help of light signals. The assumption which in this paper is presupposed below is derived in Schelb (1996) from the postulate.

1.2. Mathematical Basis

Our mathematical point of departure is determined by the axioms of EPS up to Axiom C, i.e., by a Weyl space-time. Mathematically, a Weyl manifold is a triple $(M, \mathcal{G}, \nabla^W)$ consisting of:

(a) A four-dimensional manifold M, which we assume to be of class C^∞, the topology of which is Hausdorff and second countable.
(b) A conformal equivalence class $\mathcal{G}$ of metrics g on M with Lorentzian signature $(+1, +1, +1, -1)$.
(c) A symmetric linear connection ∇^W on M with vanishing torsion.

The set of timelike geodesics of ∇^W (without specification of parametrization) constitutes the so-called *projective structure* of $(M, \mathcal{G}, \nabla^W)$; they are physically interpreted as the worldlines of freely falling particles. The null paths of $\mathcal{G}$ are interpreted as worldlines of light signals; they are ∇^W geodesics, too. The Axiom C of "compatibility" of the conformal and projective structure expresses the physical experience that freely falling particles can approximate ("chase") the worldlines of light arbitrarily closely.

1.3. The Line of Thought

The concrete purpose of this paper is the following: It formulates a certain property which any model of the EPS axiomatics can have or not, and demonstrates that the subclass of models characterized by this feature consists of *reducible* Weyl manifolds, which are equivalent to Lorentzian manifolds. Thus an improved handling of the problem described above can be achieved if this attribute can be connected to plausible and physically more intelligible (in the sense of space-time theories) statements about free fall and light propagation.

From the point of view of the mathematical structures listed above, the distinctive feature in question may appear somewhat hidden: It refers to the choice of a particular metric from the conformal class $\mathcal{G}$ of $(M, \mathcal{G}, \nabla^{W})$ and in the second place to a local property of the representation of this metric in special coordinates, namely locally geodesic ones. However, from the point of view of another, to some extent similar space-time axiomatics (Schroeter, 1988; Schroeter and Schelb, 1992, 1993; Schelb, 1992), which has served as a heuristic guideline for the present considerations, it is absolutely natural to investigate this property. If one has decided to single out a specific metric from $\mathcal{G}$, then the one which we will use is more or less uniquely determined by the available criteria. The intuitive meaning of this characteristic can be described as a certain symmetry between the conformal and projective structure; with regard to the locally geodesic coordinates the "lightcone" maintains a symmetricity along a geodesic, which in the general case is given only in one single event. This opens a new look at the relation of reducible and irreducible Weyl space-times: The former ones are distinguished by an additional symmetry of the null and timelike parts of their geodesic path families, and these mathematical structures can better be mapped into empirical statements concerning particle paths and light worldlines.

The paper is organized as follows: In Section 2.1 we sketch the way in which the conformal structure is constructed in the EPS theory. This is applied in Section 2.2 in order to pick out a special metric according to some heuristic ideas. In Section 3 we describe the construction of the special coordinate system and some crucial properties of the components of the chosen metric therein. Having these tools, we can specify which subclass of EPS models is to be investigated. Section 4 demonstrates that if one considers Lorentzian space-times in the same coordinate system, they show the characteristic property of this subclass. That means that it is indeed a *necessary* condition for a model of EPS axiomatics to be pseudo-Riemannian. Section 5 and the remainder are devoted to the proof of its sufficiency.

2. CONFORMAL CLASS AND METRIC

2.1. Construction of the EPS Conformal Structure

In order to prepare the choice of a special distinguished metric we have to sketch in which way the conformal structure is constructed in the EPS axiomatics [for formulations of the postulates and proofs see besides the original paper also Meister (1994)]. Let $\mathcal{P}_0$ denote the set of worldlines of freely falling particles and P one of its elements; let $q \in M$ be any event with $q \in P \in \mathcal{P}_0$. Let further $f\colon P \mapsto I \subset \mathbb{R}$ be an arbitrary parametrization ("clock") of this worldline P. Then (as follows from the axioms) there is a neighborhood $U(q)$ so that each $p \in U(q)\backslash P$ can in a unique manner be connected to two events $e_1, e_2 \in P$ by light signals. This can be used to define a function $h\colon U(q) \mapsto \mathbb{R}$ (which will serve as an auxiliary device for the definition of a metric) by

$$h_q(p) := -[f(e_2) - f(q)] \cdot [f(e_1) - f(q)] \tag{1}$$

e_1 is assumed to be "earlier" than e_2: $f(e_1) < f(e_2)$. By choosing the zero of the parametrization so that $f(q) = 0$, h takes the simpler form

$$h_q(p) = -f(e_2) \cdot f(e_1) \tag{2}$$

One can extend the domain of definition of h to $U \cup P$ by the stipulation that for $p \in P$ the corresponding events e_1 and e_2 are identified with p: $e_1 = p = e_2$. Then h specializes further to

$$h_q(p) = -f^2(p) \tag{3}$$

Let H_q denote the set of all events connected to q by light rays. If $p \in H_q$, then either $e_1 = q$ or $e_2 = q$, so that $h_q(p) = 0$.

One postulates (we omit the motivations) that h_q is C^2 in the argument p. On this basis one can define pointwise a metric tensor as a twofold derivation of the function h_q in the event q:

Definition. Given P, $q \in P$ and a parametrization f, and vectors Y_q, $Z_q \in T_qM$, $g_q\colon T_qM \times T_qM \mapsto \mathbb{R}$ is defined by

$$g_q(Y_q, Z_q) := Y(Z(h_q))|_q \tag{4}$$

One can prove the following properties (see, e.g., Meister, 1994):

(a) $dh_q|_q = 0$ (which warrants the tensorial transformation of the twofold derivative of h_q).
(b) $g_q(Y_q, Z_q) = g_q(Z_q, Y_q)$.
(c) If $V_q \in T_qM$ is the tangent vector of a light signal, then $g_q(V_q, V_q) = 0$ (a direct consequence of the mentioned property of h_q).

That g_q is at least of class C^2 with respect to q is given in the EPS theory by a postulate.

The definition of g in q depends on the choice of a special particle P and a special parametrization, but it can be shown that a change of P and f changes the metric g_q in q only up to a conformal factor e^{φ}. In obvious symbolic notation

$$g'_q[P', f'] = e^{\varphi(q)} \cdot g_q[P, f] \tag{5}$$

So as a result of these prescriptions one has a conformal structure $\mathcal{G}$ on M.

The following observation is important: If g_q is determined by the particle P and the parametrization f, and if X_q is the tangent vector of P with respect to f in q, then because of (3)

$$g_q(X_q, X_q) = -2 \tag{6}$$

2.2. Choice of a Special Metric

As a next step we apply the foregoing in order to select, at first locally, a special metric from $\mathcal{G}$.

Let $p_0 \in M$ be arbitrary; let P be any freely falling particle with $p_0 \in P$. We denote the worldline of P by γ; since γ is a geodesic of ∇^{W}, we can choose for its parametrization f in particular an affine parameter, denoted by t. Then, if $\dot{\gamma}$ denotes the tangent vector of γ with respect to this parameter t one knows that $\nabla^{\mathrm{W}}_{\dot{\gamma}}\dot{\gamma} = 0$.

Given γ and t, we can define in a neighborhood of p_0 the auxiliary function h_{p_0} and thus a metric g_{p_0} in p_0. The same can be done in other events $q \in \gamma$, so that along γ a special metric is constructed.

In order to expand the definition beyond the single curve γ, we consider the worldlines of all other freely falling particles passing through p_0. These, too, are timelike geodesics of ∇^{W}. Thus one can choose also ∇^{W}-affine parametrizations τ on them; denoting their curves by γ_i: $\mathbb{R} \mapsto M$ and their tangent vectors with respect to τ by $\dot{\gamma}_i$, one knows $\nabla^{\mathrm{W}}_{\dot{\gamma}_i} \dot{\gamma}_i = 0$. Affine parameters are not unique, but given only up to a constant factor. For the definition, however, we need a unique parametrization and so adapt τ to t on γ by the following condition using the already defined g_{p_0} in p_0:

$$g_{p_0}(\dot{\gamma}, \dot{\gamma}) = g_{p_0}(\dot{\gamma}_i, \dot{\gamma}_i) \tag{7}$$

The motivations and meaning of this condition in the frame of a space-time axiomatics are discussed in Schroeter (1988), Schroeter and Schelb (1992, 1993), and Schelb (1992); they are necessary if one wants to construct a metric on the basis of parametrizations of worldlines. In any case, (7) is obviously the most natural choice of the affine parameters and agrees with the usual notion of standard clocks in Lorentz manifolds.

If $J^+(p_0) \subset M$ designates the set of all events which can be connected uniquely with p_0 by particle worldlines (i.e., the "interior of the forward lightcone" H^+_{p0}), then we see that we have defined functions h_q for all these q in $J^+(p_0)$ and thus constructed a unique metric throughout $J^+(p_0) \cup p_0$.

Because of (6) we will finally define $\tilde{g} := \frac{1}{2}g$, and will in all that follows omit again the tilde. Our special distinguished metric thus has by construction in all $q \in J^+(p_0)$ the property

$$g_q(\dot{\gamma}_i, \dot{\gamma}_i) = -1 \tag{8}$$

if γ_i is the particle worldline passing through p_0 and q.

In all this the special particle γ was arbitrarily chosen. If one takes a γ' running also through p_0 and starts the metric construction with it, one gets the same metric in $J^+(p_0)$ if only one starts with the same parametrizations.

3. LOCALLY GEODESIC COORDINATES

We will work in a special coordinate system which allows a simple representation of the above constructed g. These are the well-known locally geodesic or Riemannian normal coordinates, which by means of the exponential map can be introduced at least in a neighborhood of any event of the manifold. Their most important feature is that they map all the geodesics of the used connection into straight lines through the origin. We apply them here with respect to ∇^W and its geodesics, i.e., the worldlines of freely falling particles and their above selected affine parameters. Let ψ designate the coordinate map ψ: $U \subset M \mapsto \mathbb{R}^4$. The coordinates are constructed according to the following requirements:

(a) $\psi(p_0) := (0, 0, 0, 0)$.

In T_{p0} a basis (E_1, E_2, E_3, E_4) has to be specified:

(b) One special free-fall worldline γ has to be chosen, and the timelike basis vector is defined by $E_4 := \dot{\gamma}$ so that $\psi(\gamma(t)) = (0, 0, 0, t)$ [for $q \in J^+(p_0)$, $t \geq 0$].
(c) E_1, E_2, E_3 are spacelike vectors chosen so that $g_{p0}(E_i, E_4) = 0$, $i = 1, 2, 3$, and $g_{p0}(E_i, E_j) = 0$, $i, j = 1, 2, 3$; $i \neq j$.
(d) The affine parametrizations of the non-timelike geodesics $\tilde{\gamma}$ are chosen according to $g_{p0}(\dot{\gamma}, \dot{\gamma}) = -g_{p0}(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})$ in obvious notation.

The well-known fact that in these coordinates, for the connection coefficients of ∇^W, $\Gamma^a_{bc}(p_0) = 0$, $a, b, c = 1, 2, 3, 4$, holds has no direct application in our context.

Consequences for $g_{\mu\nu}$ in these coordinates are as follows:

1. At the origin, one has $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta = \mathrm{diag}(+1, +1, +1, -1)$.
2. If the coordinates of $\dot{\gamma}$ are denoted by v^μ, $v = \psi(\dot{\gamma})$, then $g(\dot{\gamma}, \dot{\gamma}) = g_{\mu\nu}v^\mu v^\nu$. For all events on γ, i.e., on the four-axis holds $v = (0, 0, 0, 1)$, and thus we see from $g(\dot{\gamma}, \dot{\gamma}) = -1$ that
$$g_{44} = -1 \tag{9}$$
along the positive four-axis.

The coordinates are based on the choice of one special particle γ, which is mapped on the four-axis. But this particle was not distinguished by any specific property, so that in fact we have a family of coordinate systems, each one defined by the particle γ_i which yields the four-axis, all with properties 1 and 2.

Below we will use the transformations between these coordinate systems. Let us denote by $P(1, 4)$ the plane spanned by the one- and the four-axes of the coordinates: $P(1, 4) := \{x = (x^1, x^2, x^3, x^4) \,|\, x^2 = x^3 = 0\}$. Since we will consider mainly curves in $P(1, 4)$, we are interested in those transformations which leave this plane invariant. These take a very simple form, since all particle worldlines are straight lines through the origin; thus the transformations are determined by that of $T_{p0}M$.

Let us denote two coordinate systems of the above kind by ψ and ψ'; let $v, w \in T_{p0}M$ be two particle tangent vectors so that in ψ the v is mapped on the four-axis and w is in $P(1, 4)$: $\psi(v) = (0, 0, 0, 1)$, $\psi(w) = (w^1, 0, 0, w^4)$, and in the other system $\psi'(v) = (v'^1, 0, 0, v'^4)$, $\psi'(w) = (0, 0, 0, 1)$. From the manner of construction of the coordinates one reads off that such a choice is possible.

Proposition 1. The transformation of ψ into ψ' leaves the one–four plane invariant.

Proof. Let $(A_{\mu\nu})$ denote the transformation matrix in $T_{p0}M$; let $U \in T_{p0}$ so that $\psi(u) = (u^1, 0, 0, u^4)$. From $A\psi(w) = (0, 0, 0, 1)$ and $A\psi(v) = (v'^1, 0, 0, v'^4)$ it follows that $A_{21} = A_{24} = A_{31} = A_{34} = 0$ and $(A\psi(u))^i = 0$ for $i = 2, 3$. Since any $x \in P(1, 4)$ is lying on a particle worldline, from $x = \mu \cdot u$, $\mu \in \mathbb{R}$, it follows that $x' = \mu' \cdot \psi(u)$ and thus the proposition follows. ∎

The next proposition demonstrates that the vector w which is transformed into the four-axis in ψ' determines the transformation with its components in ψ. Denote $u' = \psi'(u)$.

Proposition 2. For a tangent vector of a particle worldline $u \in T_{p0}M$ with $u = (u^1, 0, 0, u^4)$ in ψ we have
$$u'^4 = u^4w^4 - u^1w^1; \qquad u'^1 = \sqrt{(u'^4)^2 - 1} \tag{10}$$

Proof. (a) Since in ψ as in ψ' one has $g_{p_0} = \eta$, the relation $-1 = g_{p0}(u, u) = g_{p0}(u', u')$ means $(u'^1)^2 - (u'^4)^2 = -1$.

(b) From $g_{p0}(u, w) = g_{p0}(u', w')$ and $w' = (0, 0, 0, 1)$ one gets $u^1w^1 - u^4w^4 = -u'^4$. ■

Note that for all tangent vectors of freely falling particles with their above selected parametrization in $P(1, 4)$ of such a coordinate system the reasoning in part (a) of the proof holds; they all obey the relation $(v^1)^2 - (v^4)^2 = -1$.

For each point y in $P(1, 4)$ of ψ which lies on a particle worldline one knows $y = \lambda u$, $\lambda \in \mathbb{R}$, if u is the corresponding tangent vector. Hence from Proposition 2 one immediately derives the components y'^μ of y in ψ' as

$$y'^1 = \lambda u'^1 = w^4y^1 - w^1y^4, \qquad y'^4 = w^4y^4 - w^1y^1 \tag{11}$$

Note that the transformations have to respect the time orientation of the particle worldlines. From (11) we get for the transformation matrix A for tangent vectors from ψ to ψ', according to $A_{\mu\nu}(y) = \partial y'^\mu/\partial y^\nu$,

$$A_{11} = w^4, \qquad A_{14} = -w^1, \qquad A_{41} = -w^1, \qquad A_{44} = w^4 \tag{12}$$

i.e., the $A_{\mu\nu}$ do not depend on y and indeed are identical to those at the origin p_0 given in Proposition 2.

For the inverse transformation one gets analogously

$$y^1 = w^4y'^1 + w^1y'^4, \qquad y^4 = w^1y'^1 + w^4y'^4 \tag{13}$$

If we denote $B := A^{-1}$, then $B_{\mu\nu} = \partial y^\mu/\partial y'^\nu$ and we obtain

$$B_{11} = w^4, \qquad B_{14} = w^1, \qquad B_{41} = w^1, \qquad B_{44} = w^4 \tag{14}$$

and similarly to Proposition 1 we know that $B_{21} = B_{24} = B_{31} = B_{34} = 0$.

Now we can consider the transformation of the components of the metric g; if $g'_{\mu\nu}$ denotes the components in ψ', then

$$g'_{\mu\nu}(y) = B_{i\mu}(y)B_{k\nu}(y)g_{ik}(y), \qquad \mu, \nu, i, k = 1, 2, 3, 4 \tag{15}$$

For tangent vectors in the one–four plane we need only the metric components $g_{11}, g_{14} = g_{41}, g_{44}$; their transformed versions can be calculated from the above:

$$g'_{11} = (w^4)^2g_{11} + 2w^1w^4g_{14} + (w^1)^2g_{44} \tag{16}$$

$$g'_{14} = w^1w^4(g_{11} + g_{44}) + g_{14}\{(w^1)^2 + (w^4)^2\} \tag{17}$$

$$g'_{44} = (w^1)^2g_{11} + 2w^1w^4g_{14} + (w^4)^2g_{44} \tag{18}$$

$$g_{11} = (w^4)^2g'_{11} - 2w^1w^4g'_{14} + (w^1)^2g'_{44} \tag{19}$$

$$g_{14} = -w^1w^4(g'_{11} + g'_{44}) + \{(w^1)^2 + (w^4)^2\}g'_{14} \tag{20}$$

$$g_{44} = (w^1)^2 g'_{11} - 2w^1 w^4 g'_{14} + (w^4)^2 g'_{44} \tag{21}$$

4. A PROPERTY OF LORENTZ SPACE-TIMES

In this section a digression away from the EPS space-times is made. We consider a general Lorentz manifold (M, g), but in the same special situation as the EPS models above. That is, we choose a point $p_0 \in M$, take the family of timelike ∇^g-geodesics which pass through p_0, and pick ∇^g-affine parametrizations on them, which are adapted to one another as described above. Then in a neighborhood U of p_0 we can construct completely analogously the locally geodesic coordinates based in p_0. We demonstrate that then we have additional information about the components of g.

Let us designate by ϕ the corresponding coordinate map ϕ: $U \subset M \mapsto \mathbb{R}^4$; $\hat{g}_{\mu\nu} = (\phi(g))_{\mu\nu}$; g is assumed to be a sufficiently smooth tensor field.

Proposition 3. In the locally geodesic coordinate representation by ϕ of a Lorentzian manifold (M, g) for all points y of the forward four-axis [i.e., $y = (0, 0, 0, y^4)$ with $y^4 \geq 0$] we have

$$\hat{g}_{14}(y) = 0 \tag{22}$$

This relation is a consequence of the Gauss lemma for pseudo-Riemannian manifolds. For the proof of the proposition we employ it in the form given by Sachs and Wu (1977): Consider a rectangle $\mathfrak{D} = [a, b] \times [-\epsilon, \epsilon] \subset \mathbb{R}^2$ with $0 < a < b$ and $\epsilon \in (0, \infty)$, which is mapped into the manifold by σ: $\mathfrak{D} \mapsto M$, $\sigma \in C^\infty$. If one keeps one of the arguments $u \in [a, b]$, $v \in [-\epsilon, \epsilon]$ of σ fixed, one has a curve in M parametrized by the other argument. Denote the corresponding vector fields T_1 and T_2, counting u as the first, v as the second argument. The Gauss lemma then requires two presuppositions:

1. For each fixed value of v the resulting curve is a geodesic; if $\sigma_v(u) := \sigma(u, v)$, then for all $v \in [-\epsilon, \epsilon]$, σ_v: $[a, b] \mapsto M$ is geodesic.
2. The geodesic vector field T_1 is of constant length with respect to the Lorentz metric: $g(T_1, T_1) = \text{const}$ in $\mathfrak{D}$.

The statement of the Gauss lemma is that along each of the geodesics σ_v the relation

$$g(T_1, T_2) = \text{const} \tag{23}$$

holds.

For our purpose here we specify σ as follows. Consider the $P(1, 4)$ plane of the coordinate system ϕ and the set of ∇^g-geodesics γ through p_0 which lie in $P(1, 4)$. Let us choose from this set all those between an arbitrary timelike geodesic and its symmetric counterpart on the other side of the four-axis; i.e., the tangent vectors of these boundary curves w, $\overline{w}$ with respect to the affine parametrizations are such that $w^1 = -\overline{w}^1$, $w^4 = \overline{w}^4$. Because of $(w^1)^2 - (w^4)^2 = -1$ one can parametrize the set of these geodesics by w^1. We write $w^1 =: v$, $w^4 = (1 + v^2)^{1/2}$ and denote the corresponding curves in M by γ_v and $\phi(\gamma_v) = \hat{\gamma}_v$. The affine parameter of any γ_v is denoted by t_v. Then we define $\sigma(u, v) := \gamma_v(t_v)$ with $t_v = u$. In the domain of definition $\mathcal{D}$ the a is understood to be small and the value of ϵ determines the boundary curves of the chosen set of geodesics. The four-axis is given by $\hat{\gamma}_0$, $\hat{\gamma}(t_0) = (0, 0, 0, t_0)$, with $\hat{T}_j = \phi(T_j)$, $j = 1, 2$; we know from the Gauss lemma that

$$\hat{g}_{\mu\nu}\hat{T}_1^\mu\hat{T}_2^\nu = \text{const} \tag{24}$$

along any of the $\hat{\gamma}_v$.

We evaluate this in particular along $\hat{\gamma}_0$, where $\hat{T}_1 = (0, 0, 0, 1)$. In order to determine $\hat{T}_2$, we have to consider the curves $\phi(\sigma_u(v))$ where u is kept fixed. With $u \equiv t_v$ they contain all points $u\cdot(v, 0, 0, (1 + v^2)^{1/2})$, $v \in [-\epsilon, \epsilon]$, the constant value of u marking the respective curve. Then

$$\hat{T}_2 = \frac{\partial}{\partial v}(\phi(\sigma_u(v))) = u\cdot\left(1, 0, 0, \frac{v}{\sqrt{1 + v^2}}\right) \tag{25}$$

and in particular on $\hat{\gamma}_0$: $\hat{T}_2 = (u, 0, 0, 0)$. Plugging this into (24) gives on $\hat{\gamma}_0$

$$\hat{g}_{41}\cdot u = \text{const} \tag{26}$$

Since here $u = t_0$, (26) means in the case const $\neq 0 \neq \hat{g}_{41}$ that for points near the origin p_0, i.e., for $u \to 0$, $|\hat{g}_{41}|$ must increase proportionally to u^{-1}. The lower limit for u is a, which cannot reach 0, but approaches it arbitrarily closely, so that there is no finite upper boundary of $|\hat{g}_{41}|$ in this process. In p_0, however, one has $\hat{g}_{41} = \eta_{41} = 0$, so that we get a contradiction to the continuity of $\hat{g}_{41}$ in p_0. Thus the only possibility to obey (26) is $\hat{g}_{41} \equiv 0$ along γ_0, which proves the proposition above.

5. CONSEQUENCES IN EPS SPACE-TIMES

We come back to the consideration of the special metric chosen from the EPS conformal class $\mathcal{G}$. The general question which we want to answer now is whether this metric for a subclass of EPS models (specified in a moment) plays a distinguished role in $\mathcal{G}$ in the following sense: Are the geodesics of ∇^W simultaneously those curves between a pair of two events (in U) which are of extremal length with respect to g? If so, they are also geodesics of the metric connection ∇^g of g.

The selection of the subclass will be made by the property which we isolated in the preceding section:

Assumption. In each coordinate system of the kind constructed above,

$$g_{14} = 0 \tag{27}$$

holds on its positive four-axis. In the locally geodesic coordinates constructed in Section 3 this leads us to the following task: Consider any two events q_1, q_2 on the positive four-axis. They are connected by the particle worldline γ_1 for which this coordinate system is constructed. If ν: $\mathbb{R} \mapsto \mathbb{R}^4$ is another timelike curve running through q_1 and q_2, one can ask the question: For any such ν, is its length with respect to g between q_1 and q_2 smaller than the corresponding length of the ∇^W-geodesic γ_1?

Let us assume in a first step that ν is inside $P(1, 4)$.

The length is independent of the parametrization of the curve; thus we use here as convenient parametrization that by the x^4 coordinate. For γ_1 mapped on the four-axis of the coordinates this is identical to its affine parameter t. If $\psi(q_1) = (0, 0, 0, x_1^4)$, $\psi(q_2) = (0, 0, 0, x_2^4)$, one asks whether the following holds:

$$\int_{x_1^4}^{x_2^4} \left\{-g\left(\frac{d\nu}{dx^4}, \frac{d\nu}{dx^4}\right)\right\}^{1/2} dx^4 \leq \int_{x_1^4}^{x_2^4} \left\{-g\left(\frac{d\gamma_1}{dx^4}, \frac{d\gamma_1}{dx^4}\right)\right\}^{1/2} dx^4 \tag{28}$$

This can be answered positively if for all x^4 with $x_1^4 \leq x^4 \leq x_2^4$,

$$g\left(\frac{d\nu}{dx^4}, \frac{d\nu}{dx^4}\right) \geq g\left(\frac{d\gamma_1}{dx^4}, \frac{d\gamma_1}{dx^4}\right) = -1 \tag{29}$$

In this relation g is compared at two *different* points (one on ν, one on γ_1) which have the same four-coordinate. The difficulty in verifying (29) is that for the point on ν we do not know the concrete $g_{\mu\nu}$; on γ_1 only $g_{44} = -1$ is of relevance.

5.1. Possibility of $g_{44} < -1$

One way to look for an answer is the following: Through each of the points of ν there passes a geodesic coming from the origin; let us denote it here by γ_i. If t_i denotes the affine parameter of γ_i specified by (7), we know $g(d\gamma_i/dt_i, d\gamma_i/dt_i) = -1$ in all of its events. For the tangent vector $u := d\gamma_i/dt_i$, at the origin one has $\eta(u, u) = \eta(d\gamma_1/dx^4, d\gamma_1/dx^4) = -1$, so that $(u^1)^2 - (u^4)^2 = -1$ along γ_i, and because of $|u^1| \neq 0$, one has $u^4 \geq 1$. On the other hand, for the tangent vector of ν, $w := d\nu/dx^4$, one knows by construction that $w^4 \equiv 1$.

Let z be the point of intersection of γ_i and ν; if in z one has $g_{44} < -1$, then (29) may be wrong: Let for instance ν in q be such that $w = (0, 0, 0, 1)$; then $g_q(w, w) = g_{44} < -1$. Thus a necessary condition for (29) is that, at any point inside the forward lightcone of p_0 in our coordinates, $g_{44} \geq -1$ (which is its value on the four-axis); that this is indeed *sufficient* for $g(w, w) \geq -1$ along v is shown in Proposition 7 below.

In this paper we are not able to give a *complete* proof for the condition $g_{44} \geq -1$, and thus below we use another reasoning. But it is very interesting to illuminate the meaning of this condition and to demonstrate what is to be done to prove it.

To this end let us consider again the coordinate transformations of the $g_{\mu\nu}$; the above assumption leads to important consequences for them. Let γ_j be any ∇^W-geodesic through the origin p_0, let v denote its tangent vector with $v^1 \neq 0$; as usual $v^2 = v^3 = 0$, $g(v, v) = -1$; and let y be any point on γ_j. If we denote all entities in the "rest system" of γ_j (i.e., where it is mapped on the four-axis) with a tilde [e.g., $\tilde{y} = (0, 0, 0, \tilde{y}^4)$], we have the following result:

Proposition 4. We have

$$g_{11}(y) = (v^4)^2 \cdot \tilde{g}_{11}(\tilde{y}) - (v^1)^2 \tag{30}$$

$$g_{14}(y) = v^1 v^4 \cdot (1 - \tilde{g}_{11}(\tilde{y})) \tag{31}$$

$$g_{44}(y) = (v^1)^2 \cdot \tilde{g}_{11}(\tilde{y}) - (v^4)^2 \tag{32}$$

Proof. Equations (30)–(32) are immediately obtained by inserting $\tilde{g}_{44}(\tilde{y}) = -1$ and $\tilde{g}_{14}(\tilde{y}) = 0$ in the general transformations (19)–(21). ∎

The meaning of the proposition is that if g_{44} is given in y, then so are g_{11} and g_{14}: From g_{44} one can calculate $\tilde{g}_{11}(\tilde{y})$ according to (32) and from this by (30) and (31), respectively, $g_{11}(y)$ and $g_{14}(y)$.

The value -1 for $g_{44}(y)$ is the borderline between different situations:

Proposition 5. (i) $g_{44}(y) \geq$ resp. < -1 means $\tilde{g}_{11}(\tilde{y}) \geq$ resp. < 1.

(ii) $\tilde{g}_{11}(\tilde{y}) \leq$ resp. > 1 means $g_{11}(y) \leq$ resp. > 1.

(iii) For $y^1 > 0$, $\tilde{g}_{11}(\tilde{y}) \leq$ resp. > 0 means $g_{14}(y) \geq$ resp. < 0; for $y^1 < 0$, $g_{14}(y) \leq$ resp. > 0.

Proof. (i) Because of $(v^1)^2 - (v^4)^2 = -1$ one can write (32) as

$$\tilde{g}_{11}(\tilde{y}) = \frac{g_{44}(y) + (v^1)^2 + 1}{(v^1)^2} = \frac{g_{44}(y) + 1}{(v^1)^2} + 1 \tag{33}$$

from which the proposition can be read off.

Parts (ii) and (iii) are obvious ($y^1 = a \cdot v^1$, $a \in \mathbb{R}^+$). ∎

The physical difference between the two situations is made visible by a further evaluation of the transformations (16)–(21); if $g_{44} < -1$, they show that at coordinate points near the null geodesics through p_0 the local light cone tips over in such a way that there are timelike worldlines which run *backward* in the time coordinate x^4. If $g_{44} \geq -1$ this phenomenon does not occur. If one can formulate an argument to exclude such awkward situations, one excludes $g_{44} < -1$.

5.2. Extremality of g_{44}

The main consequence of our Assumption rests on the following result:

Proposition 6. For points x on the four-axis, $x = (0, 0, 0, x^4)$, one has

$$2\frac{g_{14}}{x^4} + \frac{\partial g_{44}}{\partial x^1} = 0 \tag{34}$$

Proof. Let $w(x)$ denote the field of tangent vectors of the particle worldlines in $P(1, 4)$ through p_0 with respect to the chosen parametrizations. All these w obey $(w^1)^2 - (w^4)^2 = -1$, and everywhere in the domain of definition of w and g the relation $g(w, w) = -1$ holds. Hence

$$0 = \frac{\partial g(w, w)}{\partial x^1} = \frac{\partial}{\partial x^1}\{g_{11}(x)(w^1(x))^2 + 2g_{14}(x)w^1(x)w^4(x) + g_{44}(x)(w^4(x))^2\} \tag{35}$$

The dependence of w on the coordinates x is given by

$$w^1(x^1, x^4) = \frac{x^1}{\sqrt{(x^4)^2 - (x^1)^2}}, \qquad w^4(x^1, x^4) = \frac{x^4}{\sqrt{(x^4)^2 - (x^1)^2}} \tag{36}$$

Thus the derivation $\partial/\partial x^1$ yields

$$\begin{aligned} 0 = {} & \left(\frac{\partial g_{11}}{\partial x^1}\right)\frac{(x^1)^2}{(x^4)^2 - (x^1)^2} + g_{11}\frac{2x^1((x^4)^2 - (x^1)^2) + 2(x^1)^3}{[(x^4)^2 - (x^1)^2]^2} \\ & + 2\frac{\partial g_{14}}{\partial x^1}\left(\frac{x^1x^4}{(x^4)^2 - (x^1)^2}\right) + 2g_{14}\frac{x^4[(x^4)^2 - (x^1)^2] + 2(x^1)^2x^4}{[(x^4)^2 - (x^1)^2]^2} \\ & + \frac{\partial g_{44}}{\partial x^1}\frac{(x^4)^2}{(x^4)^2 - (x^1)^2} + g_{44}(x^4)^2\frac{2x^1}{[(x^4)^2 - (x^1)^2]^2} \end{aligned} \tag{37}$$

On the four-axis one has $x^1 = 0$, so that this expression reduces to (34). ∎

Since on the four-axis we have $g_{14}(x) = 0$, we get furthermore the following result.

Corollary. On the four-axis,

$$\frac{\partial g_{44}}{\partial x^1} = 0 \tag{38}$$

Thus on lines $x^4 = \text{const} \geq 0$, g_{44} has in $x^1 = 0$ an extremum. We discuss the three possibilities of a minimum, a maximum, or a saddle point, in order to show that in any case either the ∇^{W}-geodesics *are* the curves of *maximal* g-length or that they are excluded since they lead to contradictions.

5.2.1. Minimum

We have to compare the length of any timelike curve in $P(1, 4)$ connecting two points q_1, q_2 on the four-axis with the length of the corresponding segment of the axis itself. If $\psi(q_i) = (0, 0, 0, x_i^4)$, $i = 1, 2$, then one has to determine whether

$$\int_{x_1^4}^{x_2^4} \{-g(w, w)\}^{1/2}\, dx^4 < \quad \text{or} \quad > x_2^4 - x_1^4 \tag{39}$$

As discussed above, in the case where g_{44} has a minimum on the four-axis, the $<$ sign in (39) is valid:

Proposition 7. If at a point y, $g_{44}(y) \geq -1$, then for any vector $u \in T_yM$ with $u^4 = 1$, one has $g(u, u) \geq -1$.

Proof. Let v be the tangent vector of the particle worldline on which y is lying. We rewrite $g(u, u)$ so that its dependence on g_{44} is evident:

$$g(u, u) = g_{11}(u^1)^2 + 2g_{14}u^1 + g_{44} \tag{40}$$

$$= (u^1)^2\left[(v^4)^2\,\frac{(v^4)^2 + g_{44}}{(v^1)^2} - (v^1)^2\right] + 2u^1\left(v^1v^4\left(1 - \frac{(v^4)^2 + g_{44}}{(v^1)^2}\right)\right) + g_{44} \tag{41}$$

If we vary u^1 in this equation, we can see the value of it for which $g(u, u)$ becomes minimal; building the derivation of (40) with respect to u^1 leads to the result

$$u^1_{\min} = -\frac{v^1v^4(1 - \hat{g}_{11})}{(v^4)^2\cdot\hat{g}_{11} - (v^1)^2} \tag{42}$$

using the "rest system" value $\hat{g}_{11} = [(v^4)^2 + g_{44}]/(v^1)^2$ as an abbreviation. Plugging $u^1_{\min}$ again in (40) and expressing g_{44} by $\hat{g}_{11}$ yields after an elementary calculation

$$g(u_{\min}, u_{\min}) = -\frac{\hat{g}_{11}}{(v^4)^2\cdot\hat{g}_{11} - (v^1)^2} \geq -1 \tag{43}$$

because $\hat{g}_{11} \geq 1$. ∎

Thus, according to the above discussion, the axis segment is of maximal length.

5.2.2. *Maximum*

In the case where g_{44} has a maximum on the four-axis, it is not automatically clear that in (29) the $<$ and in (39) the $>$ are correct. In the subcase where in (29) the $\geq$ remains valid, again the maximal length of the axis segment is confirmed. In the other subcase where the $<$ holds, the axis segment is of *minimal* length: Assume that there are values of x^4 so that $(w^4 = 1)$

$$g(w, w) = g_{11}(w^1)^2 + 2g_{14}w^1 + g_{44} < -1 \tag{44}$$

Let us assume that a curve from q_1 to q_2 running through the region where $x^1 < 0$ is of greater length than the four-axis. In (43) from $g_{44} < -1$ it follows that $g_{14} < 0$ because of (31); the first term on the right-hand side of (43) is positive, the second depends on the sign of w^1. If one passes over to a curve which is the mirror image with respect to the four-axis, i.e., $w'^1 = -w^1$, both terms keep their sign. Thus if one considers curves very near the four-axis ($|x^1| << 1$), then in every x^4, where the relation (43) holds, it is also true for the mirror-curve ν' in the region $x^1 \geq 0$. That means that the length of ν' is greater than $x_2^4 - x_1^4$, too. Thus the four-axis is of minimal length. But by its construction according to the EPS procedure g is a *pseudo*-Riemannian metric, and we are considering *timelike* curves. So this case would yield a contradiction.

5.2.3. *Saddle Point*

The case of a saddle point of g_{44} on the four-axis proves to be very pathological. Assume, for example, that the value of g_{44} is decreasing for $x^1 \geq 0$ and increasing for $x^1 < 0$. Hence for $x^1 \geq 0$ one has $g_{44}(x^1) < -1$ and thus because of (30) $g_{11}(x^1) < 1$; for $x^1 < 0$ one has $g_{44}(x^1) > -1$ and $g_{11}(x^1) > 1$. Since $g_{11}(x^1)$ is a continuous function, $g_{11} = 1$ on the four-axis. Without this condition $g_{11} = \eta_{11}$, a saddle point is impossible. Let us consider any coordinate transformation "to the right," i.e., where x^1 is mapped into $\tilde{x}^1$ with $\tilde{x}^1 \geq x^1$. The $g_{\mu\nu}$ transformations specialize for the point on the four-axis into

$$\hat{g}_{44}(\hat{x}) = (w^1)^2 \cdot 1 - (w^4)^2 = -1, \qquad \hat{g}_{11}(\hat{x}) = (w^4)^2 \cdot 1 - (w^1)^2 = 1 \tag{45}$$

i.e., for this point the $g_{\mu\nu}$ remain unchanged. But for all points with $x^1 < 0$, one has in its "rest system" $\tilde{g}_{11} \geq 1$ and thus after the transformation $\hat{g}_{44} \geq -1$. That is, on the four-axis of the new coordinates g_{44} *has* a minimum;

analogously, it has in the case of a transformation "to the left" a maximum. Either g has to be partially an indefinite and partially a definite metric, or it has in both regions the property, that the particle worldlines through p_0 are of maximal length with the exception of that where the saddle point occurs. Thus we can exclude this case.

5.3. Non-$P(1, 4)$-Curves

Up to now we have considered only curves from q_1 to q_2 in $P(1, 4)$; of course, all preceding considerations remain true for a curve in any radial plane spanned by any spacelike vector and the four-axis, since a purely spatial rotation of the coordinate axes makes this plane into $P(1, 4)$. Accordingly we extend our assumption:

Assumption. In each coordinate system of the kind constructed above there holds on the positive four-axis

$$g_{i4} = 0, \qquad i = 1, 2, 3 \tag{46}$$

Thus the length of all such radial curves is smaller than that of the four-axis. Can a "nonradial" curve, i.e., one with $v^2 \neq 0$, $v^3 \neq 0$ for its tangent vector v, then violate this? Let us consider any point y of such a curve, rotate the coordinates so that y is in $P(1, 4)$, and decompose $v = w + u$, where $w = (w^1, 0, 0, w^4)$ is the projection of v on $P(1, 4)$. Then $g(v, v) = g(w, w) + 2g(w, u) + g(u, u)$. The last term is positive, since u is spacelike; $g(w, w) \leq -1$ if g_{44} has a minimum on the four-axis. Thus for curves near the four-axis where $|w^1|$ and also g_{24} and g_{34} are very small, the term $g(w, u)$ becomes small, and we have $g(v, v) \leq g(w, w)$. If the statement is true for all nonradial curves near the four-axis, this is enough to show that it is a geodesic of the chosen g. The extension of the argument to the case of a maximum of g_{44} is obvious.

We can summarize that the particle worldline which is mapped on the four-axis is a geodesic of the Levi-Civita connection of g; this particle was arbitrarily chosen in all preceding reasonings, so that they can be repeated with all other particle worldlines running through p_0.

6. EXTENSION OF THE METRIC

Now we direct our attention to those ∇^W-geodesics which run through $J^+(p_0)$, but not through p_0. In order to extend the definition of our distinguished metric g beyond $J^+(p_0)$ it is necessary to answer for the other ∇^W-geodesics the question of whether they are also geodesics of the metric connection of g, denoted ∇^g. This is investigated here.

Since g is taken from the conformal class $\mathscr{G}$, its ∇^g is *conformally equivalent* to ∇^W (Meister, 1994, Chapter 4). This means:

Proposition 8. For all vector fields X, $Y \in TM$ there is a $\phi_g \in TM$ determined by g so that the following equation holds:

$$\nabla_X^W Y = \nabla_X^g Y + g(\Phi_g, X)Y + g(\Phi_g, Y)X - g(X, Y)\Phi_g \tag{47}$$

For a proof consult, for instance, Meister (1994).

Let us apply this relation to any event a of the domain of definition of g in particular with $X = Y = \dot{\gamma}_i$, where γ_i is the particle worldline running through p and a:

$$\nabla_{\dot{\gamma}_i}^W \dot{\gamma}_i = \nabla_{\dot{\gamma}_i}^g \dot{\gamma}_i + 2g(\Phi_g, \dot{\gamma}_i)\dot{\gamma}_i - g(\dot{\gamma}_i, \dot{\gamma}_i)\Phi_g \tag{48}$$

By definition, $\nabla_{\dot{\gamma}_i}^W \dot{\gamma}_i = 0$ and $g(\dot{\gamma}_i, \dot{\gamma}_i) = -1$; our result that γ_i is of extremal length with respect to g yields

$$2g(\Phi_g, \dot{\gamma}_i)\dot{\gamma}_i = -\Phi_g \tag{49}$$

Thus Φ_g is proportional to $\dot{\gamma}_i$ in a: $\Phi_g(a) = r \cdot \dot{\gamma}_i|_a$, $r \in \mathbb{R}$.

Inserting this again in (47) enforces $r = 0$ and so $\Phi_g(a) = 0$. Consider now a geodesic $\tilde{\gamma}$ of ∇^W through a, but not through p_0. Applying (46) in a at the tangent vector $\dot{\tilde{\gamma}}$ of $\tilde{\gamma}$ provides

$$\nabla_{\dot{\tilde{\gamma}}}^W \dot{\tilde{\gamma}} = \nabla_{\dot{\tilde{\gamma}}}^g \dot{\tilde{\gamma}} + 2g(\Phi_g, \dot{\tilde{\gamma}})\dot{\tilde{\gamma}} - g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})\Phi_g \tag{50}$$

Because of $\Phi_g = 0$ in a and if the used parameter of $\tilde{\gamma}$ is affine so that $\nabla_{\dot{\tilde{\gamma}}}^W \dot{\tilde{\gamma}} = 0$, we get $\nabla_{\dot{\tilde{\gamma}}}^g \dot{\tilde{\gamma}} = 0$ in a. Since a was arbitrarily chosen, we can infer that everywhere in $J^+(p_0)$ the same is true for $\tilde{\gamma}$, so that it is a geodesic of ∇^g, too.

For the construction of the definition of g *beyond* $J^+(p_0)$ one has to take a ∇^W geodesic, say $\tilde{\gamma}$, passing through $J^+(p_0)$ and fix a parametrization on it. If in an event of $J^+(p_0)$ this $\tilde{\gamma}$ intersects with a γ_i, one has to apply (7) to choose its affine parameter. But can this be done in a *unique* manner?

If (7) is used in $a \in J^+(p_0)$ with a γ_1 and in $b \in J^+(p_0)$ with a γ_2, does this yield the same parametrization on $\tilde{\gamma}$? The last proposition allows an affirmative answer: It means that the number $g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})$ is constant along $\tilde{\gamma}$ for any affine parametrization: $\nabla_{\dot{\tilde{\gamma}}}^g(g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})) = 0$. Then

$$g_a(\dot{\gamma}_1, \dot{\gamma}_1) = g_a(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}) = g_b(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}) = g_b(\dot{\gamma}_2, \dot{\gamma}_2) \tag{51}$$

in consistency with

$$g_a(\dot{\gamma}_1, \dot{\gamma}_1) = g_{p_0}(\dot{\gamma}_1, \dot{\gamma}_1) = g_{p_0}(\dot{\gamma}_2, \dot{\gamma}_2) = g_b(\dot{\gamma}_2, \dot{\gamma}_2) \tag{52}$$

Thus the definition of g can be extended to all of M, and the same reasoning as above shows that all geodesics of ∇^W are geodesics of ∇^g, too. That means that ∇^g is not only conformally equivalent, but also *projectively equivalent* to ∇^W, and hence that a $(M, \mathscr{G}, \nabla^W)$ in that subclass of EPS space-times in which our assumption holds is reducible to a Lorentzian manifold.

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